# 22C:253 Lecture

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IP for metric facility locatin problem

Indicator variable $y_i \in \{0,1\}$ indicates if facility $i \in F$ is open
Indicator variable $x_{ij} \in \{0,1\}$ indicates if city $j \in C$ is connected to facility $i \in F$.

$$\text{minimize} \sum_{i \in F} y_i \cdot f_i + \sum_{i \in F, j \in C} x_{ij} \cdot c_{ij}$$

s.t.

$$\sum_{i \in F} x_{ij} \geq 1 \text{ for each } j \in C$$

$$y_i - x_{ij} \geq 0 \text{ for each } i \in F, j \in C$$

$$y_i \in \{0,1\}, x_{ij} \in \{0,1\} \text{ for each } i \in F, j \in C$$

And the LP-relaxation for this problem is obtained by replacing $y_i \in \{0,1\}, x_{ij} \in \{0,1\}$ by $y_i \geq 0, x_{ij} \geq 0,\ \forall i \in F, j \in C$

The dual problem of LP-relaxation

Indocator variables $\alpha_j, \beta_{ij}$

$$\text{maximize} \sum_{j \in C} \alpha_j$$

s.t.

$$\sum_{j \in C} \beta_{ij} \leq f_i \text{ for each } i \in F$$

$$\alpha_j - \beta_{ij} \leq c_{ij} \text{ for each } i \in F, j \in C$$

$$\alpha_j \geq 0, \beta_{ij} \geq 0 \text{ for each } i \in F, j \in C$$

Complemetary slackness conditions

For primal problem
(1) For each $i \in F, y_i > 0 \Rightarrow \sum_{j \in C} \beta_{ij} = f_i$
(2) For each $i \in F, j \in C, x_{ij} > 0 \Rightarrow \alpha_j - \beta_{ij} = c_{ij}$

For dual problem

(3) For each $j \in C, \alpha_j > 0 \Rightarrow \sum_{i \in F} x_{ij} = 1$

(4) For each $i \in F, j \in C, \beta_{ij} > 0 \Rightarrow y_i = x_{ij}$

Interpretation of these conditions is:

Assume that we have an integral primal feasible solution $(X, Y)$. This induces a set $I$ of open facilities and an assignment $\Phi$ of each city to a facility. So let $(\alpha, \beta)$ be a feasible dual solution, soppose $(X, Y)$ and $(\alpha, \beta)$ satisfy the complemetary slackness conditions, then

(1) a facility $i \in F$ is open only when the cities contribute enough towards opening the facility.

(2) a connection from $j$ to $i$ is established only if city $j$ pays enough so that after its contribution towards opening facility $i$ has been subtracted, there is enough to pay for the connection cost to $i$.

(3) for a integral solution, it is trivial

(4) if a city $j \in C$ makes a positive contribution towards opening a facility $i \in F$, and $j$ is open then j is connected to $i$.

Approximation of primal complementary slackness conditions

$$y_i > 0 \Rightarrow \frac{f_i}{3} \leq \sum_{j \in C} \beta_{ij} \leq f_i$$

$$x_{ij} > 0 \Rightarrow \frac{c_{ij}}{3} \leq \alpha_j - \beta_{ij} \leq c_{ij}$$

The following algorithm actually maintains conditions (1), (3) and (4), only (2) is relaxed as showed above.

Algorithm

Phase 1

1. $\alpha_j = 0, \beta_{ij} = 0, \forall i \in F, j \in C, I = \emptyset$
2. increase $\alpha_j$ synchronously for all cities $j \in C$

- at a point where for some edge $(i, j), \alpha_j = c_{ij}$, such a edge is said to be tight.
- once an edge $(i, j)$ becomes tight, any further increase in $\alpha_j$ implies an increase in $\beta_{ij}$ at the same rate. (so $\alpha_j - \beta_{ij} = c_{ij}$ is maintained)
- at some point for some facility $i$, $\sum_{j \in C} \beta_{ij} = f_i$, i is said to be temporarily open. (Now $\beta_{ij}$'s can no longer increase $\Rightarrow$ $\alpha_j$'s for $j$'s that have tight edges to $i$ also can not increase.)
- once a facility $i \in F$ is open, any unconnected city $j \in C$ s.t. $(i, j)$ is tight is connected to $i$, $i$ is said to be the connecting witness for $j$.

This terminates when all cities are connected.

Observation: a city $j \in C$ can make positive contributions to several facilities $\Rightarrow$ condition (4) could be violated.

Phase 2

Let $F_t$ be the set of temporarily open facilities, let $H$ be a graph with $V(H) = F_t$ and $E(H) = \{(i, i')|i \in F_t, i' \in F_t, and \exists\ j \in C : \beta_{ij} > 0$ and $\beta_{i'j} > 0\}$

Let $I$ be a maximal independent set in $H$, and this is our set of permanently open facilities. Now we define $\Phi$:

For each $j \in C$, define $X_j = \{i \in F_t|\beta_{ij} > 0\}$, and $|I \cap X_j| \leq 1$. Now:

(1) if $|I \cap X_j| = 1$, that is, there is an open facility $i$ to which $j$ makes a contribution, set $\Phi(j) = i$

(2) if city $j \in C$ has not been connected in step (1), consider $i'$, the connecting witness of j, if $i' \in I$, set $\Phi(j) = i'$

(3) if $i' \notin I$, $i'$ has some neighbor in $H$ that is open (since $I$ is a maximal independent set), let $''$ be an arbitrary open neighbor of $i'$, set $\Phi(j) = i''$