22C:253 Lecture 15

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We now have
maxf(t,s)
subject to

$$f(i,j) \leq c(i,j) \forall (i,j) \in E. - \mathbf{I}$$

$$\sum_{j:(j,i)\in E} f(j,i) - \sum_{j:(i,j)\in E} f(i,j) \leq 0 \forall i \in V. - \mathbf{II}$$

DUAL LP:
Let d_ip be the dual variable corresponding to the Type I constraint for edge (i,j).
Let P_i be the dual variable for the Type II constraint for vertex i.
We want to minimize

$$\sum_{(i,j)\in E} c(i,j) \cdot d_{ij}$$

To obtain the dual constraints let us examine the primal constraint matrix.
Hence the dual constraint corresponding to primal variable f(i,j) are

$$d_{ij} - P_i + P_j \geq 0 \forall (i,j) \in E$$

And dual constraint corresponding to f(t,s) are

$$P_s - P_t \geq 1$$

$$d_{ij} \geq 0 \forall (i,j) \in E$$

$$P_i \geq 0 \forall i \in V$$

Hence Dual LP

$$min \sum (i,j) \in E \cdot c(i,j).d_{ij}$$

subject to

$$d_{ij} - P_i + P_j \geq 0 \forall (i,j) \in E$$

$$P_s - P_t \geq 1$$

$$d_{ij} \geq 0 \forall (i,j) \in E$$

$$P_i \geq 0 \forall i \in V$$

Consider the IP obtained by replacing $d_{ij} \geq 0$ by $d_{ij} \in$ {0,1} and $P_i \geq 0$ by $P_i \in$ {0,1}
Observe that the above LP is a relaxation of this IP.
$*$ How is this IP interpretted ?
In any feasible solution $P_s = 0$ and $P_t = 0$.
Let V_0 = { i $\in$ V | $P_i = 0$ }
V_1 = { i $\in$ V | $P_i = 1$ }
In an optimal solution d_{ij} = 0 $\forall$ (i,j) $\in$ E and { (i $\in V_0$ and j $\in V_0$) or (i $\in V_1$ and j $\in V_1$)}

$$d_{ij} = 0 \forall (i,j) \in E : i \in V_0 and j \in V_1$$

$$d_{ij} = 1 \forall (i,j) \in E : i \in V_1 and j \in V_0$$

Hence the objective function is minimizing the total capacity of edges from V_1 to V_0

Primal Dual Schema For Approximation Algorithms

Consider the following approximate complementary slackness coditions:

Approximate Primal Complementary Slackness

For each j = 1,2,....., n $x_j = 0$
OR

$$\frac{C_j}{\alpha} \leq \sum_{i=1}^{m} a_{ij} \cdot y_i \leq C_j where \alpha \geq 0$$

Approximate Dual Complementary Slackness

For each i = 1,2,......, m $y_i = 0$
OR

$$\beta \cdot b_i \geq \sum_{j=1}^{n} a_{ij} \cdot x_j \geq b_i where \beta \geq 0$$

Claim: Let x and y be feasible primal and dual solutions satisfying all of the above constraints. Then

$$\sum_{j=1}^{n} c_j \cdot x_j \leq \beta \cdot \alpha \sum_{i=1}^{m} b_i \cdot y_i$$

Proof:

$$\sum_{j=1}^{n} c_j \cdot x_j \le \sum_{j=1}^{n} (\alpha \cdot \sum_{i=1}^{m} a_{ij} \cdot y_i) \cdot x_j$$

$$= \alpha \cdot \sum_{i=1}^{m} \sum_{j=1}^{n} (a_{ij} \cdot x_j) \cdot y_i$$

$$\le \alpha \cdot \beta \cdot \sum_{i=1}^{m} b_i \cdot y_i$$

Using this in Set Cover
Set Cover LP Relaxation

$$min \sum_{j=1}^{n} c(s_j) \cdot s_j$$

Subject to

$$\sum_{j:u \in S_j} x_j \ge 1 foreachelementi = 1, 2, ..., m$$

$$x_j \ge 0 foreachj = 1, 2, ..., n$$

Dual LP

$$max \sum_{i=1}^{m} y_i$$

Subject to

$$\sum_{i \in S_j} y_i \le c(S_j) foreachj = 1, 2, ..., n$$

$$y_i \ge 0 foralli = 1, 2, ..., m$$

Let us state the Approximate Complementary Slackness Conditions with $\alpha = 1$ and $\beta =$ f, where f = max. frequency of any element i.
Approximate Primal Complementary Slackness
For each set S_j , $x_j = 0$ or

$$\sum_{i \in S_j} y_i = C(S_j)$$

Approximate Dual Complementary Slackness
For each element i = 1,2,.....,m , $y_i = 0$ or

$$\sum_{j:i \in S_j}^{n} x_j \ge 1$$

If we can come up with an integral feasible solution x and a dual solution y satisfying the slackness conditions mentioned, then we get a factor-f approximation algorithm for Set Cover.

How do we find such x and y ? First let us restate the primal complementary slackness conditions: It is saying that for each set S_j , j=1,2,....,n, we cannot have x_j ¿ 0 and

$$\sum_{i \in S_j} y_i < c(S_j)$$

≡ For each set S_j , j=1,2,.....,n, if x_j ¿ 0 then

$$\sum_{i \in S_j} y_i = c(S_j)$$

1. Start with x = 0 (integral non-feasible Primal solution) and y = 0 (feasible dual solution)
2. At each step we make x more feasible maintaining integrality.
3. At each step make y more optimal.
4. At all steps approximate complementary slackness conditions are maintained.