22C:253 Lecture 14

Imran A. Pirwani

November 14, 2002

1 Primal–Dual Framework

Primal–Dual framework has its origins in the design of exact algorithms. This framework has become a highly successful algorithm design technique over the last few years. Over the last few years many classical problems such as the *Facility Location Problem* have been approximated using the primal–dual schema and the original factor approximations have improved by leaps and bounds. In this lecture, I shall describe the primal–dual framework and describe some key properties of this schema. Then, I shall provide a few well known problems that we have already discussed and then we shall write out the dual of the primal programs.

1.1 A Quick Review

An LP in standard form is written as follows:

$$\min \sum_{j=1}^{n} e_j \cdot x_j$$

such that,

$$\sum_{j=1}^{n} a_{ij} \cdot x_j \geq b_i, \text{ for } i = 1, \ldots, m$$

and,

$$x_j \geq 0, \text{ for } j = 1, \ldots, n$$

The above standard form can be written more compactly as follows:

$$\min c^T \cdot x$$

such that,

$$A \cdot x \geq b$$

and,

$$x \geq 0$$

where we are given $c \in \mathbb{R}^n, b \in \mathbb{R}^m, A \in \mathbb{R}^{m \times n}$ and we are solving the LP for $x \in \mathbb{R}^n$.
The dual of this LP is written in terms of dual variables y instead of primal variables x as follows:

$$\max \sum_{i=1}^{m} b_i \cdot y_i$$

such that,

$$\sum_{i=1}^{m} a_{ij} \cdot y_i \leq c_j, \text{ for } j = 1, \ldots, n$$

and,

$$y_i \geq 0, \text{ for } i = 1, \ldots, m$$

In compact form, the above dual LP can be written as:

$$\max b^T \cdot y$$

such that,

$$A^T \cdot y \leq c$$

and,

$$y \geq 0$$

Let us now write a familiar primal and the try to write its dual.

2 Cardinality Vertex Cover Problem as a Primal LP

Consider the CVC problem. The LP relaxation of the problem can be written as:

$$\min \sum_{i=1}^{n} x_i$$

such that,

$$x_i + x_j \geq 1, \text{ for each } \{i, j\} \in E$$

and,

$$x_i \geq 0, \text{ for } i = 1, \ldots, n$$

Now, let us try to write its **dual**:

$$\max \sum_{e \in E} y_e$$

such that,

$$\sum_{e:e \text{ incident on } i} y_e, \text{ for each } i \in V$$

and,

$$y_e \geq 0$$

2.1 An Interpretation of the above LP Dual

Consider the dual as a relaxation of an IP. To get the IP back, we replace $y_e \geq 0$ with $y_e \in \{0, 1\}$. This IP is the *Maximum Matching Problem*.

The above LP dual has a combinatorial interpretation. However, note that not all LP duals have a combinatorial interpretation.

3 Properties of Duality

This section lays the foundation of the primal dual framework. In this section, we study two key properties of duality that we state as follows:

- Weak Duality Theorem
- Strong Duality Theorem

3.1 Weak Duality Theorem

Theorem 1 *If x is a feasible solution of the primal and y is a feasible solution of the dual then,* $b^T \cdot y \leq c^T \cdot x$.

In simpler terms, the above theorem is saying that:

"Dual is a lower bound on primal and primal is an upper bound on the dual."

The figure in Figure **??** shows a pictorial view of the above.

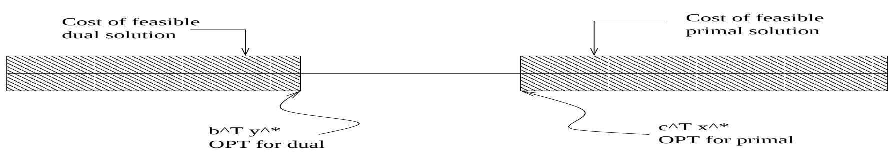

Figure 1: A Pictorial view of the Weak Duality Theorem.

Proof:

$$\begin{array}{rcl} b^T \cdot y & = & \sum_{i=1}^m b_i \cdot y_i \\ & \leq & \sum_{i=1}^m \left(\sum_{j=1}^n a_{ij} \cdot x_j \right) \cdot y_i \\ & = & \sum_{j=1}^n \left(\sum_{i=1}^m a_{ij} \cdot y_i \right) \cdot x_j \\ & \leq & \sum_{j=1}^n c_j \cdot x_j \\ & = & c^T \cdot x \end{array}$$

□

Corollary 1 *The primal has a finite optimal solution iff the dual has a finite optimal solution.*

Next, we present *Complementary Slackness Conditions* with respect to the primal and dual LP's.

3.1.1 Complementary Slackness Conditions

The "complementary slackness conditions" state the following:

> Let x and y be feasible solutions of the primal and dual LP's respectively. Then x and y are both optimal iff all the following conditions are satisfied:
>
> - **Primal Complementary Slackness Condition**. Where, for each $j = 1, \ldots, n$, either $x_j = 0$ or $\sum_{i=1}^m a_{ij} \cdot y_i = c_i$.
> - **Dual Complementary Slackness Condition**. Where, for each $i = 1, \ldots, m$ either $y_i = 0$ or $\sum_{j=1}^n a_{ij} \cdot x_j = b_i$.

3.2 Strong Duality Theorem

Theorem 2 *If x^*, y^* are primal and dual optimal solutions respectively then*

$$c^T \cdot x^* = b^T \cdot y^*$$

3.2.1 Revisiting CVC and Maximum Matching Problem in the context of the Strong Duality Theorem

Recall that the "weak duality theorem" stated that the OPT for Maximum Matching Problem (MMP) is a lower bound on the OPT for CVC. The "strong duality theorem" is stating that the OPT obtained from the relaxed version of both the problems are the same. To get a pictorial view, refer to Figure **??**.

The case of bi–partite graphs yields the picutre in Figure **??**.

The following theorem is due to König and Egervary.

Theorem 3 *The size of the a maximum matching in bi–partite graphs equals the size of a minimum* CVC.

4 The Maximum Flow–Minimum Cut Theorem

Max–Flow Problem:
Input: A digraph $G = (V, E)$ and edge capacities $c : E \to \mathbb{R}^+$ and a source s and a sink t.
Output: A flow $f : E \to \mathbb{R}^+$ such that:

1. Capacity Constraint: for each $(i, j) \in E, f(i, j) \leq c(i, j)$.

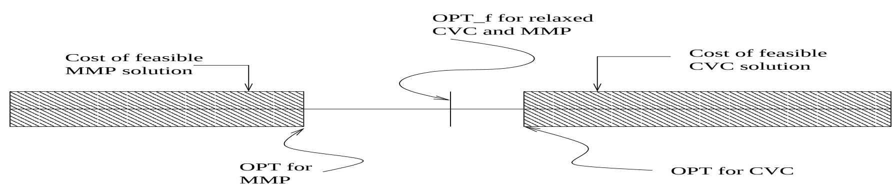

Figure 2: A Pictorial view of the Strong Duality Theorem as it applies to OPT and MMP.

2. Flow Conservation: for each $i \in V \setminus \{s, t\}$

$$\sum_{j:(j,i)\in E} f(j,i) = \sum_{j:(i,j)\in E} f(i,j)$$

3. Maximality: The following is maximized:

$$\sum_{j:(s,j)\in E} f(s,j) - \sum_{j:(j,s)\in E} f(j,s)$$

4.1 An LP for Max–Flow Problem

$$\max \sum_{j:(s,j)\in E} f(s,j) - \sum_{j:(j,s)\in E} f(j,s)$$

such that,

$$f(i,j) \leq c(i,j), \forall (i,j) \in E$$

and,

$$\sum_{j:(j,i)\in E} f(j,i) = \sum_{j:(i,j)\in E} f(i,j), \forall i \in V \setminus \{s,t\}$$

and,

$$f(i,j) \geq 0, \forall (i,j) \in E$$

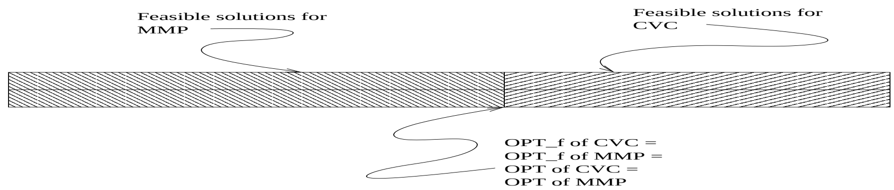

Figure 3: A pictorial view of the König–Egervary Theorem

4.1.1 A Simplification to the above LP

1. Add a directed edge (t, s) with $c(t, s) = \infty$ and enforce flow conservation at s and t. This will enable us to write out objective function as $\max f(t, s)$.
2. Replace the equality for conservation constraints by "$\leq$'s" everywhere where there are "$=$'s".

4.2 The Simplified LP

$$\max f(t, s)$$

such that,

$$f(i, j) \leq c(i, j), \forall (i, j) \in E$$

and,

$$\sum_{j:(j,i)\in E} f(j, i) - \sum_{j:(i,j)\in E} f(i, j) \leq 0, \forall i \in V$$

and,

$$f(i, j) \geq 0, \forall (i, j) \in E$$

4.3 The Dual of the above LP

- For each primal edge constraint, let p_{ij} be the corresponding dual variable.
- For each primal vertex constraint, let d_i be the corresponding dual variable.

The objective function is written as (as per the rules of the primal–dual framework):

$$\min \sum_{(i,j)\in E} p_{ij} \cdot c(i,j)$$

Now, look at the matrix representing the primal program:

$$\left(\begin{array}{cccccc}
1 & 0 & 0 & 0 & \dots & 0 \\
0 & 1 & 0 & 0 & \dots & 0 \\
0 & 0 & 1 & 0 & \dots & 0 \\
\vdots & \ddots & & & & \vdots \\
\vdots & & \ddots & & & \vdots \\
\vdots & & & \ddots & & \vdots \\
\vdots & & & & \ddots & \vdots \\
0 & 0 & 0 & 0 & \dots & 1 \\
\hline
\multicolumn{6}{c}{\dots} \\
\multicolumn{6}{c}{\ddots} \\
\multicolumn{6}{c}{+1\text{'s}, -1\text{'s and } 0\text{'s}} \\
\multicolumn{6}{c}{\ddots} \\
\multicolumn{6}{c}{\dots}
\end{array}\right)$$

In the above matrix (that corresponds to the primal LP), the submatrix above the horizontal line is an identity matrix that corresponds to the capacity constraints. The submatrix below the horizontal line contains elements in $\{0, \pm 1\}$. This submatrix corresponds to flow conservation constraints on all vertices. Since the matrix corresponds to the primal program, observe that a column corresponds to a primal variable representing an edge. The part of this column above the horizontal line will contain exactly one 1 and below the line, we will have exactly one $+1$ and exactly one -1 while the rest of the entries will be all zeros (0's).

We shall continue our discussion next time.