22C:253 Algorithms for Discrete Optimization

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We will quickly wrap up our discussion of the problem Scheduling on Unrelated Parallel Machines (SUPM) that started last lecture. We now have a solution $(T^\star, X^\star)$ to LP + (C). We showed that at most m jobs are assigned fractionally in $x^\star$. There is a simple way in which these fractional jobs can be rounded.

Consider the bipartite graph $G = (A, B, E)$ such that A is the set of machines which are assigned some fractional jobs, B is the set of jobs assigned fractionally, and E contains edges $\{i, j\}$, where $i \in A$, $j \in B$ and $x_{ij} \in (0, 1)$. This is a bipartite graph and it can be shown that this contains a matching in which all jobs are matched. This is left as an exercise for you.

As mentioned in the last lecture, in order to "round" $x^\star$ given the above property, we simply assign each job to the machine it is matched with. This increases the makespan from $T^\star$ to at most $2 \cdot T^\star \leq 2 \cdot OPT$.

Family of Tight Examples. Let $n = m^2 - m + 1$ where n is the number of jobs and m denote the number of machines. Suppose job-1, j_1, has a processing time m on all machines and any other job, j_i, can be processed in unit time on any machine.

The OPT for this problem instance is m. Say, j_1 is assigned to m_1 and completes in time m. The remaining $m^2 - m$ jobs are assigned so that each of the remaining $m - 1$ machines get m jobs. In fact, the above solution is a feasible solution for the LP. Let us consider the following feasible solution.

- Split j_1 into unit sized jobs and assign one unit to each machine.
- Of the remaining jobs, assign $(m - 1)$ of these to each machine.

This solution is a vertex of the feasibility polytope and forms a feasible solution with makespan m. If this solution is returned by the LP relaxation, then rounding will assign j_1 to one of the machines and increase the makespan to $2m - 1$.

CAPACITATED VERTEX COVER (CapVC)

INPUT: Let $G = (V, E)$ is a graph with vertex weights $w_v \in Q^+$ and vertex capacities $k_v \in Z^+$.
OUTPUT: A vertex cover defined by a function, $x : V \to N_0$ such that

(i) There is an orientation of the edges such that the number of edges coming into any vertex is atmost $k_v \cdot x(v)$.

(ii) $\sum_{v \in V} w_v \cdot x(v)$ is minimized.

Status of the Problem: A factor-2 approximation can be obtained by using dependent rounding and an alternate factor-2 approximation algorithm can be obtained using the primal–dual framework. We will discuss a simple factor-4 algorithm that uses a deterministic rounding technique and a factor-3 approximation algorithm using dependent rounding method. Given below is the integer program corresponding to CapVC. The variables used are: $x_v \in N_0$ for each $v \in V$ and $y_{e,v} \in \{0,1\}$ for each edge $e \in E$ and $v \in e$. $y_{e,v}$ indicates if vertex v covers e.

$$\text{minimize} \sum_{v \in V} w_v \cdot x_v$$

such that

$$\begin{aligned} y_{e,v} + y_{e,u} &\geq 1 \quad \text{for each edge} \quad e = \{u, v\} \in E \\ \sum_{e:v \in e} y_{e,v} &\leq k_v \cdot x_v \quad \text{for each} \quad v \in V \\ x_v &\in N_0 \\ y_{e,v} &\in \{0,1\} \end{aligned}$$

The corresponding LP relaxation replaces the constraints $y_{e,v} \in \{0,1\}$ by $y_{e,v} \geq 0$ and $x_v \in N_0$ by $x_v \geq 0$. Any feasible solution to the above IP satisfies the following property:

If $y_{e,v} = 1$ for some edge $e : v \in e$, then $x_v \geq 1$.

This property can be enforced in the LP relaxation problem by adding the following linear constraint

$$x_v \geq y_{e,v} \quad \text{for each} \quad e : v \in e$$

Deterministic Rounding Algorithm. Here is a deterministic rounding algorithm that yields a factor-4 approximation.

1. Solve the LP-relaxation to obtain the solution (X, Y).
2. For each $y_{e,v} \geq \frac{1}{2}$, $y^\star_{e,v} = 1$. For all other $y_{e,v}$, set $y^\star_{e,v} = 0$.
3. Set
$$x^\star_v = \lceil \frac{\sum_{e:v \in e} y^\star_{e,v}}{k_v} \rceil \tag{1}$$

<u>Claim</u>: This algorithm produces a factor–4 approximation algorithm.

Proof: We know that $y^\star_{e,v} \leq 2 y_{e,v} \; \forall e \in E, \; v \in e$.
And we want to show that: $x^\star_v \leq 4 x_v \quad \forall v \in V$.
Since,

$$y^\star_{e,v} \leq 2 \cdot y_{e,v}$$

we get,

$$y^\star_v = \sum_{e:v \in e} y^\star_{e,v} \;\leq\; 2 \cdot \sum_{e:v \in e} y_{e,v} \;\leq\; 2 \cdot k_v \cdot x_v \tag{2}$$

Let,

$$y_v^\star = ak_v + b \quad \forall\ a, b \in I, a \geq 0, \quad 0 \leq b \leq k_v. \tag{3}$$

So,

$$x_v \geq \frac{ak_v + b}{2k_v} = \frac{a}{2} + \frac{b}{2k_v} \tag{4}$$

Now using (1) and (3),

$$x_v^\star = \lceil \frac{y_v^\star}{k_v} \rceil \quad \leq \quad a + 1 \tag{5}$$

Therefore, if we can show that

$$(a+1) \leq 4(\frac{a}{2} + \frac{b}{2k_v}) \leq 2a + \frac{2b}{k_v}$$

we will be done.

Now, $RHS = 2a + \frac{2b}{k_v}$ If $a \geq 1$ then $RHS \geq LHS$. If $a = 0$, then $y_v^\star \leq k_v$. This implies that $x_v^\star \in \{0, 1\}$. If $x_v^\star = 0$, then we are done. If $x_v^\star = 1$, then $y_v^\star = 1$ Hence, $y_{e,v}^\star = 1$ for some edge $e : v \in e$. This implies $y_{e,v} \geq \frac{1}{2}$ for some $e : v \in e$. Therefore, $x_v \geq \frac{1}{2}$ by the constraint added to the LP relaxation. Hence, $x_v^\star \leq 4x_v$. $\square$