# 22C:253 Final Exam
# Due on Friday, 12/20 by 5 pm

1. In class we proved that the LP relaxation for VERTEX COVER (VC) has a half integral solution. We can actually prove more. First some notation. Let $G = (V, E)$ be a vertex weighted graph, and let $V_0$, $V_{1/2}$, and $V_1$ be subsets of $V$ to which an optimal solution of VC's LP relaxation assigns values 0, 1/2, and 1 respectively.

   **Theorem:** There exists an optimal solution, $V^*$ for VC such that: (a) $V^* \subseteq V_1 \cup V_{1/2}$ and (b) $V_1 \subseteq V^*$.

   Prove the above theorem.

2. The *dependent rounding* scheme described in class was randomized. Now consider the following simple deterministic variant of this:

   Set, either

   $$y_{i,j} := y_{i,j} + \alpha \text{ for all } (i,j) \in M_1 \text{ and } y_{i,j} := y_{i,j} - \alpha \text{ for all } (i,j) \in M_2$$

   or

   $$y_{i,j} := y_{i,j} - \beta \text{ for all } (i,j) \in M_1 \text{ and } y_{i,j} := y_{i,j} + \beta \text{ for all } (i,j) \in M_2$$

   depending of which makes the cost of the objective function more optimal. If both choices yield the same objective function, arbitrarily break the tie.

   The dependent rounding scheme considers the two possibilities of (i) transferring some of $y$ from $M_1$ to $M_2$ and (ii) transferring some of $y$ from $M_2$ to $M_1$ and randomizes over these. In the above version, we just pick one of the two possibilities greedily and transfer $y$ accordingly.

   Prove or disprove: if we replace the randomized depending rounding scheme by the above deterministic depending rounding scheme, then the algorithm *Threshold and Round* with the choice of threshold equal to 2/3 still yields a factor-3 approximation.

3. The CONNECTED DOMINATING SET (CDS) problem takes as input a connected undirected (unweighted) graph $G = (V, E)$ and returns a dominating set of smallest cardinality that induces a connected graph in $G$.

   Consider the following greedy algorithm for CDS. This is essentially the greedy algorithm for SET COVER.

   (a) Color all vertices white.
   (b) Pick a vertex $v$ with maximum degree, color it black.
   (c) Color all neighbors of $v$ gray.
   (e) Repeat the following two steps until all vertices are colored black or gray.
   (f) Pick a gray vertex $u$ with maximum number white neighbors.
   (g) Color $u$ black and color all of its white neighbors gray.

   At the end of the algorithm, the black vertices form a connected dominating set.

   (a) We would like to claim that the above algorithm produces an $O(\log n)$-factor approximation. Show with a counterexample that this is not true.

(b) A small modification of the above algorithm yields an $O(\log n)$-factor approximation. Define the *white degree* of a vertex $v$, denoted $degree_w(v)$, to be the number of white neighbors of a vertex $v$. For a pair of vertices $u$ and $v$, define the white degree, $degree_w(u, v)$, to be the number of white vertices that are adjacent to $u$ or $v$. Currently, in Step (f), we pick a gray vertex $u$ that maximizes $degree_w(u)$. Modify (f) so that we pick either a vertex $u$ or an edge $(u, v)$ with with maximum white degree satisfying the constraint that $u$ is gray. If, in this step, we pick a vertex $u$ we proceed as in the above algorithm. In, we pick an edge $(u, v)$, we color both $u$ and $v$ black and color any white vertex that is a neighbor of $u$ or $v$ gray. Show that with this modification the above greedy algorithm provides a factor-$O(\log n)$ guarantee.

4. The PARTIAL VERTEX COVER problem (PVC) takes as input a vertex weighted graph $G = (V, E)$ and an integer $p \geq 0$. The problem is to find a subset of vertices with smallest weight that cover (some) $p$ edges.

   (a) Write down the LP relaxation for this problem and the dual of this relaxation.

   (b) Use the primal-dual framework to obtain a factor-4 approximation algorithm for this problem.

5. Give a gap-preserving reduction from MAX-3SAT(29) to MAX-3SAT(5) with appropriate parameters to show the hardness of approximation for the latter problem.

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