Algorithmic Excursions: Topics in Computer Science II **Spring 2016**

Week 12 : Aggregating Ranking

Lecturer: Kasturi Varadarajan *Scribe: Zhe Li*

1.1 Aggregating Rankings

Central question: There is a set of items which several experts have ranked, how to aggregate these rankings?

Let's take an specific example. Assume we have three experts rank the item α, β, γ and they give the three different ranking based their knowledge as follows:

$$\pi_1 : \alpha > \beta > \gamma$$
$$\pi_2 : \beta > \gamma > \alpha$$
$$\pi_3 : \gamma > \alpha > \beta$$

The problem is that how to aggregate the rankings π_1, π_2 and π_3 together?

Firstly, let's define distance between two different rankings.

Definition 1.1 *Given permutation π and π^*, the distance between this two permutation π and π^* is*

$$d_k(\pi, \pi^*) = \#\{(i,j) : \pi^*(i) < \pi^*(j) \text{ and } \pi(j) < \pi(i)\}$$

Note: ranking can be mathematically considered as permutation. subscript "k" in $d_k(\pi, \pi^*)$ refers kemeny. Back to the previous example on items α, β, γ, based on the above definition, we have

$$d_k(\pi_1, \pi_2) = 2$$
$$d_k(\pi_2, \pi_3) = 2$$
$$d_k(\pi_3, \pi_1) = 2$$

Here, we explain the first one, for π_1: $\pi_1(\alpha) = 1, \pi_1(\beta) = 1, \pi_1(\gamma) = 3$ and for π_2: $\pi_2(\beta) = 1, \pi_1(\gamma) = 1, \pi_1(\alpha) = 3$, thus, $d_k(\pi_1, \pi_2) = \#\{(\alpha, \beta), (\alpha, \gamma)\}$. Intuitively, this distance is the number of pair of items that are ranked inversely in two different ranks.

1.1.1 Mallow's Model

Assume the true but unknown permutation π^* and parameter β, the probability of another permutation π is given

$$Pr(\pi|\pi^*) = \frac{1}{Z(\beta)} e^{-\beta d_k(\pi, \pi^*)}$$

The term $Z(\beta)$ is normalization term for probability. Let's back to our previous example again: if that true but unknown distribution π^* is π_1, then

$$Pr(\pi_1|\pi^*) \propto \frac{1}{e^{0*\beta}}$$
$$Pr(\pi_2|\pi^*) \propto \frac{1}{e^{2*\beta}}$$

Table 1.1: Pair of items

	Yes	No
$\alpha > \beta$	2	1
$\beta > \gamma$	2	1
$\gamma > \alpha$	2	1

1.1.2 Mallow's Reconstruction Problem

After discussing the above generating assumption, it will be easy to construct as follows: Given the different ranks $\pi_1, \pi_2, \cdots, \pi_r$, find that permutation π^* that maximizes the probability that generates the given r permutations:

$$\prod_{k=1}^{r} Pr(\pi_k|\pi^*) = \frac{1}{Z(\beta)^r} e^{-\beta \sum_{k=1}^{r} d_k(\pi_k, \pi^*)}$$

or equivalently minimize

$$\sum_{k=1}^{r} d_k(\pi_k, \pi^*)$$

For notation purpose, we define $d(\pi^*) := \sum_{k=1}^{r} d_k(\pi_k, \pi^*)$. In order to compute $d(\pi^*)$, we can narrow down to pair of items. For the previous example on α, β, γ, we can have

Claim 1.2 *Without the generating assumption, Mallow's Reconstruction Problem (MRP) is NP-Hard.*

With high probability, MRP can be solved in polynomial time. The procedure of solving MRP in polynomial time is following:

- For each item α, compute the average rank
$$\bar{\pi}(\alpha) = \frac{\sum_{k=1}^{r} \pi_i(\alpha)}{r}$$
- Argue with high probability that we have
$$|\pi^{opt}(\alpha) - \bar{\pi}(\alpha)| = O(\log n)$$
for each item α
- Given such $\bar{\pi}(\alpha)$ for each item α, we can compute π^{opt} in polynomial time.

Let's analyze the first argument. We prove that by giving the two step high-level idea:

- argue that $|\bar{\pi}(k) - \pi^*(k)|$ is small, small in the sense of $\log(n)$
- argue that $|\pi^m(k) - \pi^*(k)|$ is also small, in the sense of $\log(n)$.

For proving the first argument, we use the following lemma, which can be considered as the procedure of how to generate random permutation given the optimal permutation.

Lemma 1.3 *Let α be the element ranked k^{th} by π^*, then for a generated π*

$$Pr(|\pi(\alpha) - k| \geq i) < ce^{-\beta i}$$

where $\pi^*(\alpha) = k$

Assume items are $1, 2, \cdots, n$ and $\pi^*(i) = i$. Imagine π be generated as follows:

- Suppose ordering of item $1, 2, \cdots, j-1$ has been generated; For example, now we have already generated $\{3, 1, 4, 2\}$
- Place item j in spot $j - i$ with probability proportional to $e^{-\beta i}$; back to the generated $\{3, 1, 4, 2\}$, we would like to place 5 in the sequence, the probability that we put 5 in the spot as shown in Table 1.2:

Table 1.2: Probability for putting 5 in the different spot

Spot for 5	Prob
5	$e^{-\beta*0}$
4	$e^{-\beta*1}$
3	$e^{-\beta*2}$
2	$e^{-\beta*3}$
1	$e^{-\beta*4}$

Claim 1.4 *With high probability, for any i and j such that $j \geq i + L$, $\pi_l(j) \geq \pi_l(i)$ for at least $\frac{2}{3}r$ of permutations $\pi_1, \pi_2, \cdots, \pi_r$*

The above claim implies that with high probability for every i and j such that $j \geq i + L$, $\pi^m(j) > \pi^m(i)$. Note that: readers to this note can refer Braverman and Mossel [1] for details.

References

[1] Mark Braverman and Elchanan Mossel. Sorting from noisy information. *CoRR*, abs/0910.1191, 2009. URL `http://arxiv.org/abs/0910.1191`.